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FINITENESS OBSTRUCTIONS FOR HOMOLOGICALLY NILPOTENT SPACES

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The Wall finiteness obstruction $w(X)$, when X is any finitely dominated homologically nilpotent complex with finite fundamental group, is shown to lie in the kernel class group $D(\mathbb{Z}[\pi_1(X)]) \subseteq \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$.

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A topological space X is called *homologically nilpotent* if the homology groups $H_i(\tilde{X})$ ($\tilde{X}$ the universal cover) all are nilpotent as $\mathbb{Z}[\pi_1(X)]$-modules. Here, a $\mathbb{Z}[G]$-module M is called nilpotent if there is a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ by submodules, such that G acts trivially on each quotient M_i/M_{i-1}. Mislin and Varadarajan [3] showed that if X is a finitely dominated (homologically) nilpotent space, and if $\pi_1(X)$ is a finite nilpotent group, then the finiteness obstruction $w(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ actually lies in the subgroup $D(\mathbb{Z}[\pi_1(X)])$. The main result here is to extend this result to the case of an arbitrary finite fundamental group.

For any finite group G, $D(\mathbb{Z}G)$ is defined as the kernel

$$D(\mathbb{Z}G) = \mathrm{Ker}[\tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathfrak{M})],$$

when $\mathfrak{M} \subseteq \mathbb{Q}G$ is any maximal order containing $\mathbb{Z}G$. Being a maximal order, $\mathfrak{M}$ has the property that any finitely generated torsion free $\mathfrak{M}$-module is projective [4, Corollary 21.15]. In particular, if M is any finitely generated $\mathfrak{M}$-module, then there is a short exact sequence

$$0 \to P_1 \to P_2 \to T \to 0$$

of $\mathfrak{M}$-modules, where P_1 and P_2 are finitely generated and projective. This allows the definition of an element

$$[M] = [P_2] - [P_1] \in K_0(\mathfrak{M})$$

which is independent of the choice of resolution.

Lemma 1. *Fix a group G, let $\mathfrak{M} \supseteq \mathbb{Z}G$ be a maximal order in $\mathbb{Q}G$, and let $\alpha: \tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathfrak{M})$ be the induced homomorphism. Then, for any finitely dominated CW complex X with fundamental group G and finiteness obstruction $w(X) \in \tilde{K}_0(\mathbb{Z}G)$, $H_*(X; \mathfrak{M})$ is finitely generated and*

$$\alpha(w(X)) = \sum_{i=0}^{\infty} (-1)^i [H_i(X; \mathfrak{M})] \in \tilde{K}_0(\mathfrak{M}).$$

Proof. Let $C_*(\tilde{X})$ denote the cellular chain complex for the universal covering space $\tilde{X}$ of X. By [6, Theorem F], there is a finitely generated projective $\mathbb{Z}G$-chain complex P_* which is $\mathbb{Z}G$-chain homotopy equivalent to $C_*(\tilde{X})$, and such that

$$w(X) = \sum_{i=0}^{\infty} (-1)^i [P_i] \in K_0(\mathbb{Z}G).$$

Hence $H_*(X; \mathfrak{M}) \cong H_*(\mathfrak{M} \otimes_{\mathbb{Z}G} P_*)$ is finitely generated, and

$$\alpha(w(X)) = \sum_{i=0}^{\infty} (-1)^i [\mathfrak{M} \otimes_{\mathbb{Z}G} P_i] = \sum_{i=0}^{\infty} (-1)^i [H_i(\mathfrak{M} \otimes_{\mathbb{Z}G} P_*)]$$

$$= \sum_{i=0}^{\infty} (-1)^i [H_i(\mathfrak{M} \otimes_{\mathbb{Z}G} C_*(\tilde{X}))]$$

$$= \sum_{i=0}^{\infty} (-1)^i [H_i(X; \mathfrak{M})] \in \tilde{K}_0(\mathfrak{M}). \qquad \square$$

The next lemma will be useful when studying the groups $(H_*(X; \mathfrak{M}))$ of Lemma 1.

Lemma 2. *Let X be a homologically nilpotent space with finite fundamental group G. Fix a (right) $\mathbb{Z}G$-module M and a normal subgroup $H \lhd G$ such that $\sum_{h \in H} xh = 0$ for all $x \in M$ (note that this holds if $M^H = 0$). Then, for each $i \geq 0$, $H_i(X; M)$ is a torsion group, and has p-power torsion only for primes $p \mid |H|$. In particular, if $H_1, H_2 \lhd G$ are two subgroups of relatively prime order and $M^{H_1} = M^{H_2} = 0$, then $H_*(X; M) = 0$.*

Proof. Set $\sigma = \sum_{h \in H} h \in \mathbb{Z}G$, and $n = |H|$. Fix $j \geq 0$, and let

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = H_j(\tilde{X})$$

($\tilde{X}$ the universal cover of X) be $\mathbb{Z}G$-submodules such that the G-action N_i / N_{i-1} is trivial for all $1 \leq i \leq k$. Then $(n - \sigma) N_i \subseteq N_{i-1}$ for all i, so that

$$(n-\sigma)^k H_j(\tilde{X}) = (n^k - n^{k-1}\sigma) H_j(\tilde{X}) = 0. \tag{1}$$

Now let F_* be a free $\mathbb{Z}G$-resolution of $H_j(\tilde{X})$. Using (1), we see that

$$n^{k-1}(n-\sigma): \ F_* \to F_*$$

is chain homotopic to zero. Since $M\sigma=0$ by hypothesis,

$$n^k = n^{k-1}(n-\sigma)\otimes \mathrm{id} = \mathrm{id}\times n^{k-1}(n-\sigma)\simeq 0:\ M\otimes_{\mathbb{Z}G}F_* \to M\otimes_{\mathbb{Z}G}F_*.$$

Hence, for all i,

$$n^k \cdot \mathrm{Tor}_i(M, H_j(\tilde{X})) = n^k \cdot H_i(M\otimes_{\mathbb{Z}G}F_*) = 0. \tag{2}$$

Consider the spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i(M, H_j(\tilde{X})) \Rightarrow H_*(X; M).$$

This can be derived, for example, from the double complex whose jth row is $M\otimes_{\mathbb{Z}G}PR_*(C_j(\tilde{X}))$, where $C_*(\tilde{X})$ is the cellular (or singular) chain complex, and PR_* is some functor assigning a projective resolution to each $\mathbb{Z}G$-module (and preserving exact sequences). By (2), each $E^2_{i,j}$ is a torsion group, having p-power torsion only for primes $p\,|\,|H|$. The same statement then holds for each $E^\infty_{i,j}$ and each $H_i(X; M)$; and this proves the lemma. □

One more lemma is needed. The following is well known, and proven in [3], but not stated explicitly there.

Lemma 3. *Fix a prime p. For each $n\geqslant 0$, set $\xi_n = \exp(2\pi i/p^n)\in\mathbb{C}$.*

(i) *If G is a p-group, and A is a simple summand of $\mathbb{Q}G$, then*

(a) *if p is odd, $A\cong M_r(\mathbb{Q}(\xi_n))$ for some $r\geqslant 1$ and some $n\geqslant 0$,*

(b) *if $p=2$, $A\cong M_r(D)$, where $r\geqslant 1$, and for some n, $D\subseteq \mathbb{Q}(\xi_n)$ or $D\cong \mathbb{Q}(\xi_n, j)(\subseteq H)$.*

(ii) *Fix a finite dimensional simple $\mathbb{Q}$-algebra A, and a maximal $\mathbb{Z}$-order $\mathfrak{M}\subseteq A$. Assume that for some $r\geqslant 1$ and some n, either $A\cong M_r(F)$ for some subfield $F\subseteq\mathbb{Q}(\xi_n)$, or $p=2$ and $A\cong M_r(\mathbb{Q}(\xi_n, j))$. Then, for any finite $\mathfrak{M}$-module T of p-power order, $[T]=0\in K_0(\mathfrak{M})$.*

Proof. (i) In [5, Section 2], Roquette shows that the division algebra for any irreducible $\mathbb{Q}G$-representation M is isomorphic to the division algebra of a primitive faithful representation of some subquotient of G. In [5, Section 3], he shows that the only p-groups with primitive faithful representations are the cyclic groups; and, if $p=2$, the dihedral, semidihedral, and quaternionic groups.

(ii) Write $A\cong M_r(D)$, where $D\subseteq\mathbb{Q}(\xi_n)$ or $D\cong\mathbb{Q}(\xi_n, j)$. Any maximal order in A is Morita equivalent to any maximal order in D [4, Corollary 21.17], and Morita equivalences send modules of p-power order to modules of p-power order. So it suffices to prove this when $\mathfrak{M}\subseteq A=D$. It clearly suffices to consider irreducible $\mathfrak{M}$-modules; i.e., modules of the form $\mathfrak{M}/I$ where I is a maximal left ideal.

If $D\subseteq\mathbb{Q}(\xi_n)$, then $(1-\xi_n)\subseteq\mathbb{Z}[\xi_n]$ is the unique maximal ideal of p-power index, and $N_{\mathbb{Q}(\xi_n)/D}(1-\xi_n)$ generates the unique maximal ideal of p-power index in $\mathfrak{M}$. If $D=\mathbb{Q}(\xi_n, j)$, then there is a unique maximal 2-sided ideal $J\subseteq\mathfrak{M}$ of 2-power index [4, Theorem 22.4], and $\mathfrak{M}/J$ is simple by [4, Theorem 22.3]. Assume $\mathfrak{M}$ is chosen such that $\alpha = 1-\xi\in\mathfrak{M}$. Then $\mathfrak{M}\alpha$ has index 4 in $\mathfrak{M}$ and is easily seen to be maximal;

$\mathfrak{M}/\mathfrak{M}\alpha$ is thus the unique irreducible $\mathfrak{M}/J$-module, and the unique irreducible $\mathfrak{M}$-module of 2-power index. $\square$

Using the results in [4, Sections 22, 24], Lemma 3(ii) can be extended to include any D containing a maximal subfield $E \subseteq D$, such that $E \subseteq \mathbb{Q}(\xi_n)$ for some n.

The main theorem can now be proven. In the proof, C_n, for any $n \geq 1$, will denote a (multiplicative) cyclic group of order n.

Theorem 4. *For any finitely dominated homologically nilpotent space X with finite fundamental group G, $w(X) \in D(\mathbb{Z}G)$.*

Proof. Let $\tilde{X}$ denote the universal covering space of X. By [2, Theorem 4], $D(\mathbb{Z}G)$ is preserved under induction and restriction maps. Hence, $\tilde{K}_0(\mathbb{Z}G)/D(\mathbb{Z}G)$ is detected by restriction to hyperelementary subgroups of G. For any $H \subseteq G$, the restriction (transfer) map sends $w(X) \in \tilde{K}_0(\mathbb{Z}G)$ to $w(\tilde{X}/H) \in \tilde{K}_0(\mathbb{Z}H)$; and $\tilde{X}/H$ is also homologically nilpotent. So it suffices to prove the theorem when G is hyperelementary.

Fix p such that H is p-hyperelementary. Then $G = H \rtimes \pi$, where π is a p-Sylow subgroup of G, $H \lhd G$, H is cyclic, and $p \nmid m = |H|$. For any prime $q \mid m$, H_q denotes the q-Sylow subgroup of H (and G).

Write

$$\mathbb{Q}G = \prod_{s=0}^{k} A_s,$$

where each A_s is simple, and A_0 has trivial G-action. Let $\mathfrak{M}_s \subseteq A_s$, for all s, be maximal orders such that

$$\mathbb{Z}G \subseteq \mathfrak{M} = \prod_{s=0}^{k} \mathfrak{M}_s.$$

Recall that $H_*(X; \mathfrak{M})$ is finitely generated by Lemma 1. We will show that for each $1 \leq s \leq k$,

$$\sum_{i=0}^{\infty} (-1)^i [H_i(X; \mathfrak{M}_s)] = 0 \in K_0(\mathfrak{M}_s). \tag{1}$$

The corresponding result for $\mathfrak{M}_0 \cong \mathbb{Z}$ is trivial; and so by Lemma 1, (1) implies that $w(X) \in D(\mathbb{Z}G)$.

Fix $1 \leq s \leq k$. The proof of (1) splits into two cases.

Case 1. Assume first that H acts nontrivially on A_s. In other words, if we identify

$$\mathbb{Q}H \cong \prod_{d \mid m} \mathbb{Q}(\zeta_d), \quad \mathbb{Q}G \cong \prod_{d \mid m} \mathbb{Q}(\zeta_d)[\pi]^t \qquad \zeta_d = \exp(2\pi i/d) \tag{2}$$

($H \cong C_m$, $G = H \rtimes \pi$), then A_s is a summand of $\mathbb{Q}(\zeta_{d(s)})[\pi]^t$ for some $d(s) \neq 1$.

Step 1A. Assume $d(s)$ is not a prime power. If $q_1 q_2 \mid d(s)$, where $q_1 \neq q_2$ are primes, then H_{q_1} and H_{q_2} each act with zero fixed point set on A_s ($\subseteq \mathbb{Q}(\zeta_{d(s)})[\pi]^t$). By Lemma 2, $H_*(X; \mathfrak{M}_s) = 0$, and (1) holds.

Step 1B. Now assume $d(s)=q^a$ for some prime q and some $a \geq 1$. Then $A_s^{H_q}=0$, and so $H_*(X;\mathfrak{M}_s)$ is a q-group by Lemma 2. Let $C(H_q)=C_G(H_q)$ denote the centralizer (H_q the q-Sylow subgroup). Then $C(H_q)=H_q \times K$ for some $K \lhd G$, and $q \nmid |K|$.

If $A_s^K=0$, then $H_*(X;\mathfrak{M}_s)_{(q)}=0$ by Lemma 2. But $H_*(X;\mathfrak{M}_s)$ is a q-group, and so $H_*(X;\mathfrak{M}_s)=0$.

If $A_s^K \neq 0$, then K fixes A_s ($K \lhd G$ and A_s is simple). Referring to (2), this shows that A_s is a summand of $\mathbb{Q}(\zeta_{q^a})[\pi/\rho]$, where

$$\rho=\pi \cap K=\pi \cap C(H_q)=\operatorname{Ker}[\pi \xrightarrow{\text{conj.}} \operatorname{Aut}(H_q)].$$

Since $\operatorname{Aut}(H_q)$ and $\operatorname{Gal}(\mathbb{Q}(\zeta_{q^a})/\mathbb{Q})$ have the same p-power torsion, this shows that π/ρ acts effectively on $\mathbb{Q}(\zeta_{q^a})$. Hence, by [4, Theorem 29.6, 29.8], $\mathbb{Q}(\zeta_{q^a})[\pi/\rho]^t$ is simple, and

$$A_s \cong \mathbb{Q}(\zeta_{q^a})[\pi/\rho]^t \cong M_{|\pi/\rho|}(F) \quad F=\operatorname{Fix}(\pi/\rho, \mathbb{Q}(\zeta_{q^a})).$$

For all i, $H_i(X;\mathfrak{M}_s)$ is a q-group, and $[H_i(X;\mathfrak{M}_s)]=0 \in K_0(\mathfrak{M}_s)$ by Lemma 3(ii). So (1) is proved in Case 1.

Case 2. Now assume that H fixes A_s. Since $s \geq 1$, by assumption, $A_s^G=0$; and by Lemma 2, $H_*(X;\mathfrak{M}_s)$ is a torsion group, with torsion only at primes dividing $|G|$. Since G/H is a p-group and A_s is a summand of $\mathbb{Q}[G/H]$, $[H_i(X;\mathfrak{M}_s)_{(p)}]=0$ in $K_0(\mathfrak{M}_s)$ by Lemma 3.

To prove (1), it remains to show that

$$\sum_{i=0}^{\infty} (-1)^i [H_i(X;\mathfrak{M}_s)_{(q)}]=0 \in K_0(\mathfrak{M}_s) \tag{5}$$

For any prime $q \mid m$ such that $H_*(X;\mathfrak{M}_s)_{(q)} \neq 0$. Fix such a q.

Write $K=C(H_q)$ (the centralizer). Then $K=H_q \times K'$, where $K' \lhd G$ and $q \nmid |K|$. Since $H_*(X;\mathfrak{M}_s)_{(q)} \neq 0$ by assumption, Lemma 2 implies that $A_s^{K'} \neq 0$. So K', and hence K, fix A_s. In other words,

$$A_s \text{ is a summand of } \mathbb{Q}[G/K] \quad K=C(H_q) \supseteq H. \tag{6}$$

Step 2A. Since G acts nontrivially on A_s, (6) implies that $H_q \not\subseteq Z(G)$, and so $[\pi, H_q] \neq 1$ ($G=H \rtimes \pi$). Thus, π is a p-group acting nontrivially under conjugation on H_q. In particular,

$$p \,\big|\, |[\operatorname{Ker}[\operatorname{Aut}(H_q) \twoheadrightarrow \operatorname{Aut}(H_q/[\pi, H_q])]]|.$$

Since H_q is a cyclic q-group, this shows that $[\pi, H_q]=H_q$.

By assumption, G acts nilpotently on $H_*(\tilde{X})$. In particular, any subgroup of G of order prime to q acts trivially on $H_*(\tilde{X};\hat{\mathbb{Z}}_q)$. Since π acts trivially, so does $[\pi, H_q]=H_q$; and hence G acts trivially on $H_*(\tilde{X};\hat{\mathbb{Z}}_q)$.

Step 2B. Set $p^a=|G/K|$ ($K=C(H_q) \supseteq H$), and $q^b=|H_q|$. Note that G/K is cyclic: $G/K \subseteq \operatorname{Aut}(H_q) \cong (\mathbb{Z}/q^b)^*$. Also, since $K=H_q \times K'$ where $q \nmid |K'|$,

$$H^*(K;\hat{\mathbb{Z}}_q) \cong H^*(H_q;\hat{\mathbb{Z}}_q) \quad \text{and} \quad H_*(K;\hat{\mathbb{Z}}_q) \cong H_*(H_q;\hat{\mathbb{Z}}_q). \tag{7}$$

Consider the action of G/K on $H^*(K;\hat{\mathbb{Z}}_q)$ induced by conjugation. Fix a generator $g\in G/K$. Then g acts on $H^2(K;\hat{\mathbb{Z}}_q)\cong\mathbb{Z}/q^b$ via multiplication by some unique primitive p^a-th root of unity $\zeta\in\hat{\mathbb{Z}}_q$ (note that $(\hat{\mathbb{Z}}_q)^*$ and $(\mathbb{Z}/q^b)^*$ have the same p-power torsion).

Any $\hat{\mathbb{Z}}_p[G/K]$-module M decomposes as a sum

$$M=\bigoplus_{t=0}^{p^a-1} {}^tM;$$

where for each t,

$${}^tM=\{x\in M: gx=\zeta^t\cdot x\}\subseteq M.$$

For example, by the choice of ζ, for all $0\leq t<p^a$,

$${}^tH^*(K;\hat{\mathbb{Z}}_q)=\bigoplus_{i=0}^{\infty} H^{2ip^a+2t}(K;\hat{\mathbb{Z}}_q).$$

(see (7)). Similarly, for any finitely generated $\hat{\mathbb{Z}}_q$-module M with trivial G-action, and any $1\leq t\leq p^a-1$,

$${}^tH_*(K;M)=\bigoplus_{i=1}^{\infty}[H_{2ip^a-2t-1}(K;M)\oplus H_{2ip^a-2t}(K;M)]. \tag{8}$$

The point here is that there are decompositions

$$\hat{\mathbb{Q}}_q[G/K]=\bigoplus_{t=0}^{p^a-1}\hat{\mathbb{Q}}_q^{(t)} \quad\text{and}\quad \hat{\mathbb{Z}}_q[G/K]=\bigoplus_{t=0}^{p^a-1}\hat{\mathbb{Z}}_q^{(t)}; \tag{9}$$

where for each t, g acts on $\hat{\mathbb{Q}}_q^{(t)}$ and $\hat{\mathbb{Z}}_q^{(t)}$ via multiplication by ζ^t (recall $G/K=\langle g\rangle$). If we regard A_s as a summand of $\mathbb{Q}[G/K]$ (see (6)), then

$$\hat{\mathbb{Q}}_q\otimes A_s=\bigoplus_{t\in T}\hat{\mathbb{Q}}_q^{(t)} \quad\text{and}\quad \hat{\mathbb{Z}}_q\otimes\mathfrak{M}_s=\bigoplus_{t\in T}\hat{\mathbb{Z}}_q^{(t)}, \tag{10}$$

where (recall $\mathfrak{M}_s^G=0$):

$$T=\{t: 1\leq t\leq p^a-1, {}^t(\hat{\mathbb{Z}}_q\otimes_{\mathbb{Z}}\mathfrak{M}_s)\neq 0\}.$$

In particular,

$$\sum_{t\in T}[(\mathbb{Z}/q)^{(t)}]=[\mathfrak{M}_s/q\mathfrak{M}_s]=0\in K_0(\mathfrak{M}_s); \tag{11}$$

and for any finite $\mathfrak{M}_s$-module M of q-power order:

$$[M]=\sum_{t\in T}[{}^tM]=\sum_{t\in T}(\log_q|{}^tM|)\cdot[(\mathbb{Z}/q)^{(t)}]\in K_0(\mathfrak{M}_s). \tag{12}$$

Step 2C. By (9) and (10), as $\mathfrak{M}_s$-modules,

$$\begin{aligned} H_*(X;\mathfrak{M}_s)_{(q)}&\cong H_*\Big(X;\bigoplus_{t\in T}\hat{\mathbb{Z}}_q^{(t)}\Big)\cong\bigoplus_{t\in T}{}^tH_*(X;\hat{\mathbb{Z}}_q[G/K])\\ &\cong\bigoplus_{t\in T}{}^tH_*(\tilde{X}/K;\hat{\mathbb{Z}}_q). \end{aligned}\tag{13}$$

We now consider the spectral sequence

$$E^2_{i,j} = H_i[K;\, H_j(\tilde{X}; \hat{\mathbb{Z}}_q)) \Rightarrow H_*(\tilde{X}/K; \hat{\mathbb{Z}}_q)$$

for the fibration $\tilde{X} \to \tilde{X}/K \to BK$ [1, Section XVI.9].

Cap product makes E^r_{*j}, for each $r \geq 2$ and $j \geq 0$, into a module over $H^*(K; \hat{\mathbb{Z}}_q)$; and the differentials in the spectral sequence are all $H^*(K; \hat{\mathbb{Z}}_q)$-linear. Fix any α generating $H^2(K; \hat{\mathbb{Z}}_q) \cong \mathbb{Z}/q^b$ (see (7)). Since G acts trivially on $H_j(\tilde{X}; \hat{\mathbb{Z}}_q)$ for any $j \geq 0$ (Step 2A), we get using (8) that for any $1 \leq t \leq p^a - 2$,

$$\bigcap \alpha:\ {}^tE^2_{*j} = {}^tH_*(K;\, H_j(\tilde{X}; \hat{\mathbb{Z}}_q)) \to {}^{t+1}H_*(K;\, H_j(\tilde{X}; \hat{\mathbb{Z}}_q)) = {}^{t+1}E^2_{*,j}$$

is an isomorphism. Hence, $(\bigcap \alpha)$ sends ${}^tE^r$ isomorphically to ${}^{t+1}E^r$ for all $2 \leq r \leq \infty$; and so

$$\bigcap \alpha:\ {}^tH_*(\tilde{X}/K; \hat{\mathbb{Z}}_q) \xrightarrow{\cong} {}^{t+1}H_*(\tilde{X}/K; \hat{\mathbb{Z}}_q) \tag{14}$$

is an isomorphism for all $1 \leq t \leq p^a - 2$.

By (12) and (13),

$$\sum_{i=0}^{\infty} (-1)^i [H_i(X; \mathfrak{M}_s)_{(q)}] = \sum_{t \in T} d_t \cdot [(\mathbb{Z}/q)^{(t)}] \in K_0(\mathfrak{M}_s),$$

where for each $t \in T$,

$$d_t = \sum_{i=0}^{\infty} (-1)^i \cdot \log_q |{}^tH_i(X; \mathfrak{M}_s)_{(q)}| = \sum_{i=0}^{\infty} (-1)^i \cdot \log_q |{}^tH_*(\tilde{X}/K; \hat{\mathbb{Z}}_q)|.$$

Since the isomorphism in (14) preserves $\mathbb{Z}/2$-grading, we have $d_t = d_1$ for all $1 \leq t \leq p^a - 1$, and in particular for all $t \in T$. So using (11),

$$\begin{aligned}\sum_{i=0}^{\infty} (-1)^i [H_i(X; \mathfrak{M}_s)_{(q)}] &= d_1 \cdot \sum_{t \in T} [(\mathbb{Z}/q)^{(t)}] \\ &= d_1 \cdot [\mathfrak{M}_s / q\mathfrak{M}_s] = 0 \in K_0(\mathfrak{M}_s).\end{aligned}$$

This proves (5), and finishes the proof of Case 2. □

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